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Variational Analysis in Pressure Coordinates

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by

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Table of Contents

1. The variational integral and constraint

The idea is to explicitly introduce some form of balance equation as a constraint on hemispheric or global analyses. In particular, the Flattery type of analysis ignores the nonlinear relation between wind and geopotential found in the gradient wind relation or in the balance equation. We can do this by means of a variational analysis, following the ideas of Sasaki (J. Met. Soc. Japan, 1958, vol. 36, No. 3, pp. 1-88) and Stephens (J. App. Meteor., 1970, vol. 9, pp. 737-739). In this procedure, we change an initial analysis of geopotential and streamfunction, denoted by ϕ^o and ψ^o, to a final analysis ϕ^f, ψ^f:

$$\phi^f = \phi^o + \phi',$$
$$\psi^f = \psi^o + \psi'.$$

The final fields satisfy a dynamic constraint and the spatially integrated squares of ϕ' and ψ' are minimized.

Consider K pressure surfaces at which ϕ and ψ are given, $k = 1,\ldots,K$. We can form K-1 "temperatures"

$$T_k = \gamma_k (\phi_k - \phi_{k-1}) \,, \quad k = 2, \cdots, K$$

$$\gamma_k = [R \ln (P_{k-1}/P_k)]^{-1}. \tag{1.1}$$

As the constraint, we take the balance equation

$$\nabla^2 \phi - \nabla \cdot f \nabla \psi + \nabla \cdot \left[\tfrac{1}{2} \nabla (\nabla \psi)^2 - \nabla^2 \psi \, \nabla \psi \right] = 0, \tag{1.2}$$

and linearize it to the extent of evaluating the nonlinear term only with the original streamfunction ψ^0 . Thus

$$\nabla^2 \phi' - \nabla \cdot f \nabla \psi' + \nabla \cdot \underset{\sim}{A}^0 = 0,$$

$$\underset{\sim}{A}^0 = \nabla \phi^0 - f \nabla \psi^0 + \frac{1}{2} \nabla (\nabla \psi^0)^2 - \nabla^2 \psi^0 \nabla \psi^0. \qquad (1.3)$$

$\nabla \cdot \underset{\sim}{A}^0$ is a known quantity, measuring the extent to which ψ^0 and ϕ^0 do not satisfy (1.2).

The variational problem is set up now as the problem of minimizing the area integral of

$$I = \sum_{k=1}^{K} \gamma_k (\nabla \psi_k')^2 + \sum_{k=2}^{K} \beta_k (T_k')^2 + \alpha (\phi_1')^2$$

$$+ 2 \sum_{k=1}^{K} \rho_k \left[\nabla^2 \phi_k' - \nabla \cdot f \nabla \psi_k' + \nabla \cdot \underset{\sim}{A}^0_k \right]. \qquad (1.4)$$

In this expression $T_k' = \tau_k (\phi_k' - \phi_{k-1}')$, and the variables $\alpha = \alpha(x,y)$, $\beta = \beta(x,y,p)$, and $\gamma = \gamma(x,y,p)$ are positive analysis weights. $\rho(x,y,p)$ is a Lagrange multiplier (as yet unknown). The weights α , β and γ are set inversely proportional to the assumed error in the corresponding original fields

$$\alpha \propto (\delta \phi_1^0)^{-2},$$

$$\beta \propto (\delta T^0)^{-2},$$

$$\gamma \propto (\delta \nabla \psi^0)^{-2}. \qquad (1.5)$$

For example, a highly accurate T^o field corresponds to a small δT^o, a large β, and a tendency to keep T' small in the process of minimizing (1.4).

Before passing on to the variational details, it can be pointed out that the brutal step of ignoring ψ' in the vector $\underset{\sim}{A}$ of (1.3)--which avoids the usual balance equation ellipticity problem--can be softened somewhat if desired by iterating the entire analysis cycle one or more times with ψ^o and ϕ^o replaced by successive values of ϕ^f and ψ^f.

2. Variational algebra

Let δ denote changes in ϕ_1'. The corresponding change in I is twice

$$[-\beta_2 T_2' \tau_2 + \alpha \phi_1']\delta + \rho_1 \nabla^2 \delta$$

$$= [-\beta_2 \tau_2^2 (\phi_2' - \phi_1') + \alpha \phi_1' + \nabla^2 \rho_1]\delta + \nabla \cdot [\rho_1 \nabla \delta - \delta \nabla \rho_1].$$

The divergence term integrates out on the sphere. On the Northern Hemisphere it vanishes if, at the equator,

$$\text{I)} \qquad \rho_1 = 0 \quad \text{and} \quad \delta \phi_1' = 0, \qquad (2.1a)$$

or,

$$\text{II)} \qquad \partial \rho_1 / \partial \theta = 0 \quad \text{and} \quad \partial(\delta \phi_1')/\partial \theta = 0. \qquad (2.1b)$$

An extremum with respect to $\delta \phi_1'$ therefore requires that

$$(\alpha + \beta_2 \tau_2^2)\phi_1' - \beta_2 \tau_2^2 \phi_2' + \nabla^2 \rho_1 = 0. \qquad (2.2a)$$

Now let δ denote changes in ϕ_k' , k = 2,...,K. Calculations similar

to those above lead to

For k = 2,...,K-1:

$$-\beta_k \tau_k^2 \phi_{k-1}' + (\beta_k \tau_k^2 + \beta_{k+1} \tau_{k+1}^2) \phi_k' - \beta_{k+1} \tau_{k+1}^2 \phi_{k+1}' + \nabla^2 \rho_k = 0 .$$

(2.2b)

For k = K:

$$-\beta_K \tau_K^2 \phi_{K-1}' + \beta_K \tau_K^2 \phi_K' + \nabla^2 \rho_K = 0 ,$$

(2.2c)

with boundary conditions equivalent to (2.1) on ρ_k and $\delta \phi_k'$.

Finally, let δ denote a variation in ψ_k' . The corresponding

variation of I is twice

$$\tau_k \nabla \psi_k' \cdot \nabla \delta - \rho_k \nabla \cdot f \nabla \delta =$$
$$= -[\nabla \cdot \tau_k \nabla \psi_k' + \nabla \cdot f \nabla \rho_k] \delta$$
$$+ \nabla \cdot [\tau_k \delta \nabla \psi_k' + f(\delta \nabla \rho_k - \rho_k \nabla \delta)].$$

The divergence term disappears on integration for the globe. If we treat

only the Northern Hemisphere, its vanishing requires at the equator,

I) $\quad \partial \psi_k' / \partial \theta = 0,$ (2.3a)

or,

II) $\quad \delta \psi_k' = 0$ (2.3b)

leaving the equation

$$\nabla \cdot \tau_k \nabla \psi_k' + \nabla \cdot f \nabla \rho_k = 0 .$$

(2.4)

Equations (2.2a), (2.2b), (2.2c), (2.4) and the balance equation (1.3) constitute the system of 3K equations to determine ψ', ϕ', and ρ at $k = 1,\ldots,K$ given a knowledge of α, β, γ and $\nabla \cdot \mathbf{A}^{0}$.

When the Northern Hemisphere alone is considered, the set of equatorial boundary conditions in (2.1b) and in (2.3b) are most suitable, since ψ as an odd function of latitude and ϕ as an even function of latitude is most realistic. In this case ρ is an even function also as far as the boundary condition (2.1) is concerned. For simplicity and definiteness we can in the hemispheric case consider

$$\phi, \rho, \alpha, \beta, \gamma \quad \text{as even functions of latitude, and}$$
$$\psi \quad \text{as an odd function.} \tag{2.5}$$

The balance equation (1.3) is consistent with these and poses no further restriction (except that the hemispheric integral of $\nabla \cdot \mathbf{A}^{0}$ vanish). Equations (2.2a)-(2.2c), when integrated over the hemisphere, place constraints on hemispheric integrals of $\beta_k \phi'_k$:

$$\iint \beta_k \phi'_k \, d\,hem = 0 \tag{2.6}$$

Hemispheric integration of (2.4) shows that the line integral

$$\int_{0}^{2\pi} \left[\gamma_k \frac{\partial \psi'_k}{\partial \theta} \right]_{\theta=0} d\lambda = 0. \tag{2.7}$$

If $\mathscr{V}_k$ were independent of longitude at the equator, this would be equivalent to constraining the horizontally averaged vorticity to be preserved:

$$\iint \nabla^2 \psi' \, d\,hem \;=\; \int_0^{2\pi} \left(\frac{\partial \psi'}{\partial \theta}\right)_{\theta=0} d\lambda \;.$$

3. Separation of variables

A considerable step toward making solution of the set (1.3), (2.2a)-(2.2c), and (2.4) tractable is to simplify the distribution of analysis weights. We shall assume

$$\alpha(x,y) = 2 \;\; times \;\; \beta'(x,y),$$

$$\beta(x,y,p) = \overline{\beta}_k \;\; times \;\; \beta'(x,y), \tag{3.1}$$

$$\gamma(x,y,p) = \overline{\gamma}_k \;\; times \;\; \gamma'(x,y),$$

where the horizontal functions are nondimensional and positive.

To arrive at this specification, define

$$k = 1: \qquad \mathscr{V}_1 = \frac{p_1 - p_2}{2\,(\delta w_1^o)^2} ,$$

$$\tag{3.2}$$

$$k = 2,\dots,K: \qquad \mathscr{V}_k = \frac{p_{k-1} - p_{k+1}}{2\,(\delta w_k^o)^2} ,$$

$$\beta_k = \frac{p_{k-1} - p_k}{(\delta T_k^o)^2} ,$$

where the numerators express a weighting with the pressure thickness associated with the wind and temperature values (p = press(mb) ÷ 1000).

Given these three-dimensional fields we choose $\bar{\gamma}_k$ and $\bar{\beta}_k$ to be the horizontal averages of (3.2), and define γ' and β' by

$$\gamma'(x,y) = K^{-1} \sum_{k=1}^{K} (\gamma_k / \bar{\gamma}_k),$$

$$\beta'(x,y) = (K-1)^{-1} \sum_{k=2}^{K} (\beta_k / \bar{\beta}_k).$$

(3.3)

For α , we first define a field

$$\alpha = (p_1 - p_2) / (\delta\phi_1^0)^2,$$

(3.4a)

where $\delta\phi_1^0 (x,y)$ is the analysis error in ϕ^0 at k = 1. $\bar{\alpha}$ is obtained then by

$$\bar{\alpha} = \text{spatial average of } (\alpha/\beta')$$

(3.4b)

$(\delta u)^2$ and $(\delta T)^2$ vary by a factor of about 10 from data-rich areas to the data-poor oceanic areas which occupy most of the area. γ' and β' will therefore vary from a minimum of (say) 0.2 over the oceans to 2 or 3 over the continents, but will each have a horizontal average of one. If α in (3.4a) were equated to β_2 , we would have $\delta\phi_1 = R \, \delta T_2 \, \ln (p_1/p_2)$.

$\delta\phi_1$ is also equivalent to $(R\bar{T}/p)$ times δp (sea-level). For $p_1 = 1$ and $p_2 = .85$, a δT_2 of 0.6 deg corresponds to $\delta\phi_1 = 26.6 \text{ m}^2\text{sec}^{-2}$ and δp(sea level) = 0.34 mb. We may therefore expect approximate equality of $\bar{\alpha}$ with $\bar{\beta}_2$, since 0.6 deg and .34 mb are both realistic for good analysis regions.

It is convenient to change to nondimensional variables and operators at this point, using the radius of the earth (a) and (1/2Ω) (Ω = the earth's angular velocity) as the length and time units. To be specific we set

<u>Nondimensional</u>   <u>Dimensional</u>

$$\phi_k \quad = \quad \phi_k' \sqrt{\bar{\tau}_k} / 2\Omega a$$

$$\psi_k \quad = \quad \psi_k' \sqrt{\bar{\tau}_k} / a$$

(3.5)

$$\kappa_k \quad = \quad \rho_k \, 2\Omega / a \sqrt{\bar{\tau}_k}$$

$$\nabla \quad = \quad a \, \nabla$$

The square roots of $\bar{\tau}_k$ are introduced to produce symmetry of the matrix $B_{\ell k}$ in (3.6c) below.

Equation (1.3) becomes

$$\nabla^2 \phi_k - \nabla \cdot \sin\theta \, \nabla \psi_k = -(a \sqrt{\bar{\tau}_k} / 2\Omega) \nabla \cdot \underset{\sim}{A}_k^\circ = -A_k . \tag{3.6a}$$

Equation (2.4) becomes

$$\nabla \cdot \gamma' \nabla \psi_k + \nabla \cdot \sin\theta \, \nabla \kappa_k = 0 . \tag{3.6b}$$

The K equations (2.2a), (2.2b), and (2.2c) become

$$\nabla^2 \kappa_k + \beta' \sum_{\ell=1}^{K} B_{\ell k} \phi_\ell = 0 . \tag{3.6c}$$

$B_{\ell k}$ is the symmetric tri-diagonal matrix

$$\begin{pmatrix} d_1 & -a_2 & 0 & 0 & & & & 0 \\ -a_2 & d_2 & -a_3 & 0 & & & & \\ 0 & -a_3 & d_3 & -a_4 & & & & \\ & & & & & & & \\ & & & & -a_{k-1} & d_{k-1} & -a_K & \\ & & & & & & 0 & \\ 0 & & & & & 0 & -a_K & d_K \end{pmatrix} \qquad (3.7)$$

in which the diagonal terms are

$$d_1 = \left[4\Omega^2 a^2 (\bar{\alpha} + \bar{\beta}_2 \,\zeta_2^2) \right] / \bar{\gamma}_1 \,,$$

$$d_{k=2,\cdots,K-1} = \left[4\Omega^2 a^2 (\bar{\beta}_k \,\zeta_k^2 + \bar{\beta}_{k+1}\,\zeta_{k+1}^2) \right] / \bar{\gamma}_k \,,$$

$$d_K = \left[4\Omega^2 a^2 \bar{\beta}_K \,\zeta_K^2 \right] / \bar{\gamma}_K \,, \qquad (3.8a)$$

and the off-diagonal terms are

$$a_{k=2,\cdots,K} = 4\Omega^2 a^2 \bar{\beta}_k \,\zeta_k^2 \left(\bar{\gamma}_k \,\bar{\gamma}_{k-1} \right)^{-1/2}. \qquad (3.8b)$$

$B_{\ell k}$ has K positive eigenvalues λ_j and orthogonal eigenvectors e_{kj}: √

$$\sum_k B_{\ell k}\, e_{kj} = \sum_k B_{k\ell}\, e_{kj} = \lambda_j \, e_{\ell j} \,,$$

$$\sum_k e_{kj}\, e_{k\ell} = \delta_{j\ell} \qquad (3.9)$$

√It can be shown that

$$\lambda_j \sum_{k=1}^{K} \bar{\gamma}_k \, f_k^2 = 4\Omega^2 a^2 \left[\bar{\alpha} + \sum_{k=2}^{K} \bar{\beta}_k \,\zeta_k^2 (f_k - f_{k-1})^2 \right]$$

where

$$f_k = e_{kj} \left(\bar{\gamma}_k \right)^{-1/2}.$$

We expand in these eigenvectors:

$$(H_j , S_j , R_j , F_j) = \sum_k e_{kj} (\phi_k , \psi_k , \varkappa_k , A_k) , \qquad (3.10a)$$

$$(\phi_k , \psi_k , \varkappa_k , A_k) = \sum_j e_{kj} (H_j , S_j , R_j , F_j) . \qquad (3.10b)$$

$$(j = 1, \cdots , K ; \quad k = 1, \cdots , K)$$

Equations (3.6a)-(3.6c) now become a set of K independent two-dimensional problems (j = 1,...,K):

$$\nabla^2 H_j - \nabla \cdot \sin \theta \, \nabla S_j = - F_j , \qquad (3.11a)$$

$$\nabla \cdot \gamma' \nabla S_j + \nabla \cdot \sin \theta \, \nabla R_j = 0 , \qquad (3.11b)$$

$$\nabla^2 R_j + \beta' \lambda_j H_j = 0 . \qquad (3.11c)$$

These are the equations which must be solved in the horizontal coordinates. Variable coefficients on the left side consist of $\sin \theta$, γ', and β', λ_j being a constant for each j.

Table I shows the values of λ_j and e_{kj} which exist for the case in which the K pressure levels are those of the 12 standard pressure levels p = 1., 0.85, 0.7, 0.5, 0.4, 0.3, 0.25, 0.2, 0.15, 0.1, 0.07, and 0.05, with uniform values at all k of $(\delta T)^2 = (0.6 \text{ deg})^2$ and $(\delta v)^2 = (2.5 \text{ m sec}^{-1})^2$, and with $\delta\phi_l$ equal to 26 $m^2 sec^{-2}$. (These values are probably typical of analysis errors in data-rich regions.)

Table I. Values of λ_j and e_{k_j} for 12 standard pressure levels. $(\delta T)^2 = (0.6 \text{ deg})^2$, $(\delta v)^2 = (2.5 \text{ m sec}^{-1})^2$ and $(\delta\phi_1)^2 = (26 \text{ m}^2\text{sec}^{-2})^2$. Computed with double precision arithmetic on an IBM 360/195, but e_{k_j} values are truncated for brevity.

p		1	2	3	4	5	6
	λ_j:	107.83	368.76	925.37	1705.2	2716.9	3798.9
1.00	e_{1_j}	.020	.048	.078	.099	.135	.126
.85	e_{2_j}	.057	.135	.213	.262	.343	.308
.70	e_{3_j}	.104	.238	.344	.372	.397	.270
.50	e_{4_j}	.179	.367	.396	.209	-.121	-.398
.40	e_{5_j}	.199	.360	.256	-.046	-.339	-.283
.30	e_{6_j}	.240	.346	.032	-.321	-.277	.239
.25	e_{7_j}	.234	.276	-.105	-.324	-.046	.309
.20	e_{8_j}	.285	.239	-.275	-.265	.257	.158
.15	e_{9_j}	.356	.137	-.441	.038	.444	-.366
.10	e_{10_j}	.412	-.099	-.360	.521	-.247	-.134
.07	e_{11_j}	.389	-.277	-.003	.253	-.357	.467
.05	e_{12_j}	.522	-.548	.443	-.349	.215	-.176

p		7	8	9	10	11	12	
	λ_j:	4593.9	6600.7	8860.6	10911.7	13948.9	32308.1	
1.00	e_{1_j}	.057	.081	.136	.343	.008	.895	
.85	e_{2_j}	.135	.174	.261	.586	.010	-.439	
.70	e_{3_j}	.090	.022	-.130	-.643	-.022	.075	
.50	e_{4_j}	-.241	-.398	-.398	.286	.063	-.005	
.40	e_{5_j}	-.058	.256	.658	-.179	-.185	.616	D-3
.30	e_{6_j}	.254	.392	-.314	.022	.517	-.629	D-4
.25	e_{7_j}	.146	-.148	-.213	.080	-.748	.124	D-4
.20	e_{8_j}	-.130	-.582	.370	-.066	.358	-.172	D-5
.15	e_{9_j}	-.317	.445	-.153	.020	-.075	.130	D-6
.10	e_{10_j}	.562	-.152	.031	-.003	.008	-.541	D-8
.07	e_{11_j}	-.599	.061	-.007	.514 D-3	-.001	.251	D-9
.05	e_{12_j}	.177	-.012	.001	-.559 D-4	.849 D-4	-.867	D-11

[The determinant of $B_{k\ell}$ is readily shown to equal

$$det\ B_{k\ell} = (2\Omega a)^{2K} \frac{\bar{\alpha} \cdot \bar{\beta}_2 \zeta_2^2 \cdot \bar{\beta}_3 \zeta_3^2 \cdots \bar{\beta}_K \zeta_K^2}{\bar{\tau}_1 \cdot \bar{\tau}_2 \cdot \bar{\tau}_3 \cdots \bar{\tau}_K}$$

Setting $\bar{\alpha}$ equal to zero--equivalent to ignoring $\alpha\,(\phi_1')^2$ in the integrand (1.4)--would therefore introduce one zero value for λ_j . This solution in (3.11) has zero R_j and S_j--showing that all adjustment in (3.11a) for this degenerate mode occurs in the geopotential field H_j. The corresponding eigenvalue e_{k_1} is equal to $(\bar{\tau}_k / \sum \bar{\tau}_k)^{1/2}$.]

λ_j in Table I evidently acts as the square of a vertical wave number, since the number of sign changes in each eigenvector increases steadily from zero to eleven as λ_j increases monotonically with j. [The last eigenvector is unique in its rapid fall off in magnitude for k > 1. To the extent that it resembles a delta function δ_{k_1} , it implies that equations (3.11) could almost be written directly for ϕ_k , ψ_k , γ_k and A_k at level k = 1, with $\lambda_j = \lambda_{12}$. This isolation seems undesirable. In this example $\delta\phi_1$ was chosen to correspond to a somewhat optimistic sea-level pressure error of only 1/3 mb. Larger values of $\delta\phi_1$ (smaller $\bar{\alpha}$) will reduce the semi-isolation of layer k = 1 that is present in Table I, and should be considered carefully.]

The set of equations (3.11) is easily manipulated into the following relation

$$\gamma'\,(\nabla S)_j^2 + \beta'\,\lambda_j\cdot H_j^2 = R_j \cdot F_j + \nabla \cdot \left[\sin\theta\,(S\nabla R - R\nabla S) + \gamma' S\nabla S + R\nabla H - H\nabla R \right]_j , \quad (3.12)$$

showing that the horizontally integrated modal squared amplitude is
determined by the horizontal average of RF. The equivalent statement
for the original variables ψ and ϕ' is best derived from (1.3), (2.2a),
(2.2b), (2.2c), and (2.4). This can be done without the separation
assumption (3.1) for the weights α, β, γ. (2.4) is multiplied by
$-\psi'_k$ and (1.3) is multiplied by ρ_k to produce, for each k, the relation

$$\gamma_k (\nabla \psi'_k)^2 - \phi'_k \nabla^2 \rho_k = \rho_k \nabla \cdot \underset{\sim}{A}^o_k$$

$$+ \nabla \cdot \left[f(\psi' \nabla \rho - \rho \nabla \psi') + \gamma \psi' \nabla \psi' + \rho \nabla \phi' - \phi \nabla \rho \right]_k . \tag{3.13}$$

If b_k temporarily denotes $\beta_k \gamma_k^2$, equations (2.2a), (2.2b) and (2.2c)
can be manipulated into the form

$$- \phi'_1 \nabla^2 \rho_1 = - \phi'_1 \left[b_2 (\phi'_2 - \phi'_1) - \alpha \phi'_1 \right],$$

$$- \phi'_k \nabla^2 \rho_k = - \phi'_k \left[b_{k+1} (\phi'_{k+1} - \phi'_k) - b_k (\phi'_k - \phi'_{k-1}) \right] \quad (k = 2, \cdots, K-1),$$

$$- \phi'_K \nabla^2 \rho_K = \phi'_K \left[b_K (\phi'_K - \phi'_{K-1}) \right],$$

When these are introduced into (3.13) and the result summed over k, we obtain

$$\sum_{k=1}^{K} \gamma_k (\nabla \psi'_k)^2 + \alpha (\phi'_1)^2 + \sum_{k=2}^{K} \beta_k (T'_k)^2$$

$$= \sum_{k=1}^{K} \rho_k \nabla \cdot \underset{\sim}{A}^o_k + \mathcal{D}_k \tag{3.14}$$

where $\mathcal{D}_k$ indicates a collection of $\nabla \cdot$ terms which disappear on
horizontal integration. This equation, like its specialized modal counter-

part (3.12), shows how the Lagrange multiplier ρ implicitly, through

its organization with respect to $\nabla \cdot \underline{A}^o$, acts to determine the resulting

minimum integrated value of the original integrand (1.4).

4. <u>Response to a localized distribution of $\nabla \cdot \underline{A}^o$</u>

The nature of the solutions of (3.10) and (3.11) can be investigated

under certain simplified conditions. The most fruitful simplification is

to treat $\sin\theta$ as a constant in (3.11). At the same time, let us for

convenience define

$$h_j = H_j ,$$
$$\ell_j = \sin\theta \; S_j ,$$
$$\nu_j = \sin^2\theta \; R_j , \tag{4.1}$$
$$\sigma_j = \sin\theta \sqrt{\lambda_j} .$$

(3.11) then becomes

$$\nabla^2 h_j - \nabla^2 \ell_j = - F_j ,$$
$$\nabla \cdot \gamma' \nabla \ell_j + \nabla^2 \nu_j = 0 , \tag{4.2}$$
$$\nabla^2 \nu_j + \beta' \sigma_j^2 h_j = 0 .$$

We will use these, in combination with (3.10), to discuss two effects.

Effects of irregularities in β' and γ' are discussed in section 5. In

this section we determine the response to a localized "point source" of

the forcing function $\nabla \cdot \mathbf{A}^{o}$. We do this under the further assumption

that $\beta' = \gamma' = 1$. Equations (4.2) then reduce to

$$\nabla^2 h_j - \nabla^2 a_j = -F_j \, ,$$

$$\nabla^2 a_j - \sigma_j^2 h_j = 0 , \tag{4.3}$$

which combine into

$$\nabla^2 h_j - \sigma_j^2 h_j = -F_j \, . \tag{4.4}$$

We see already that the equation for each j has associated with it

a characteristic length L_j . In dimensional units this is

$$L_j = \frac{a}{\sigma_j} = \frac{a}{\sin \theta \sqrt{\lambda_j}} \, .$$

For sin θ = 0.707, the eigenvalues of Table I produce values of L_j

ranging from L_{12} = 50 km to L_1 = 868 km. In other words, the highly

oscillatory vertical structure represented by $e_{h,12}$ in Table I, if

excited at a particular horizontal location (x_0, y_0), will disappear

rapidly with distance horizontally from (x_0, y_0). The smooth vertical

field $e_{h,1}$ in Table I, however, will extend considerably further in

the horizontal. This arrangement of response seems desirable and

suggests that the form for I in (1.4) is satisfactory.

We consider now the circulatory symmetric case (with x denoting the radial coordinate) produced by

$$0 \leq \varkappa < \varkappa_0 \quad : \quad F_j = F_{j0} = \text{constant}$$

$$\varkappa_0 < \varkappa \leq \infty \quad : \quad F_j = 0.$$

(4.5)

The differential equation is

$$\frac{1}{\varkappa} \frac{d}{d\varkappa}\left(\varkappa \frac{dh_j}{d\varkappa}\right) - \sigma_j^2 h_j = -F_j ,$$

and the solution that maintains continuity of h_j and $dh_j/d\varkappa$ at $\varkappa = \varkappa_0$, and which disappears as $\varkappa \to \infty$, is

$$\varkappa < \varkappa_0 : \quad h_j = \left[1 + U_j(\varkappa)\right] F_{j0}/\sigma_j^2$$

$$\varkappa > \varkappa_0 : \quad h_j = W_j(\varkappa) \, F_{j0}/\sigma_j^2$$

(4.6a)

where

$$U_j(\varkappa) = -\sigma_j \varkappa_0 \, K_1(\sigma_j \varkappa_0) \, I_0(\sigma_j \varkappa) ,$$

$$W_j(\varkappa) = \sigma_j \varkappa_0 \, I_1(\sigma_j \varkappa_0) \, K_0(\sigma_j \varkappa) .$$

(4.6b)

I_0, I_1, K_0 and K_1 are the modified Bessel functions described on pages 374-379 of Abramowitz and Stegun (1964).✓ U_j is negative with a maximum magnitude of 1 at $\varkappa = \varkappa_0$, $\sigma_j \varkappa_0 \to 0$. W_j is positive. Its maximum value (reached at $\varkappa = \varkappa_0$) increases from 0 to 0.5 as $\sigma_j \varkappa_0$ varies from 0 to ∞ . The general character of the solution (4.6a) is a simple bell-shaped curve, having a discontinuity in $d^2h/d\varkappa^2$

✓Handbook of Mathematical Functions. National Bureau of Standards, June 1964.

at $\mathcal{N}_0$. ✓

Let us define the original forcing function as having the special form

$$\frac{\nabla \cdot \underset{\sim}{A}_0}{f^2} \quad = \text{constant} = \left[\frac{(\nabla \cdot \underset{\sim}{A}^0)_\ell}{f^2}\right]_{\text{center}} \tag{4.7}$$

for $\mathcal{N} < \mathcal{N}_0$ at one level, $k = \ell$, zero everywhere at all other levels and also zero at level ℓ for $\mathcal{N} > \mathcal{N}_0$. If $\nabla^2 \phi^0 \sim f \nabla^2 \psi^0$, $\nabla \cdot \underset{\sim}{A}^0$ has the order of magnitude (velocity/horizontal scale length)^2 and it will tend to be negative in cyclonic flow, positive in anticyclonic flow. We might expect (4.7) to be as large as $-(.25)^2$ at the tropopause in a major middle-latitude trough region, and perhaps even closer to one in magnitude in sharp curved jets.

From (3.10b) and (3.6a) we then find that

$$F_j \ (\mathcal{N} > \mathcal{N}_0) = 0 , \tag{4.8}$$

$$F_j \ (\mathcal{N} < \mathcal{N}_0) = F_{j0} = \frac{f^2 a}{2\Omega} e_{\ell j} \left(\frac{\bar{\tau}}{\ell}\right)^{1/2} \left(\frac{\nabla \cdot \underset{\sim}{A}^0}{f^2}\right)_\ell ,$$

✓ For $\mathcal{N}_0 \to 0$, $w_j \to \frac{1}{2} (\sigma_j \mathcal{N}_0)^2 K_0(\sigma_j \mathcal{N})$, suggesting an alternate formulation in which $F_{j0} \mathcal{N}_0^2$ is kept fixed, but $\mathcal{N}_0$ is set equal to zero:

$$h_j = \frac{1}{2} \left[F_{j0} \mathcal{N}_0^2\right] K_0(\sigma_j \mathcal{N}) , \quad 0 < \mathcal{N} \leq \infty .$$

This has a logarithmic singularity at $\sigma_j \mathcal{N} \to 0$. This is a true "point source," but seems less satisfactory for a quantitative interpretation.

where our choice of the single level at which $\nabla \cdot \underset{\sim}{A}^{o}$ is not zero

prescribes the value of ℓ. This is consistent with (4.5).

As measures of the response to this localized forcing, we first

consider the change in geostrophic vorticity and the change in actual

vorticity, both nondimensionalized through a division by f. We first find,

at level k, that

$$\frac{1}{f^{2}} \nabla_{dim}^{2} (\phi^{f} - \phi^{o})_{k} = \frac{2\Omega}{a f^{2} \sqrt{\bar{F}_{k}}} \sum_{j} e_{k_{j}} \cdot \nabla^{2} \lambda_{j} ,$$

$$\frac{1}{f} \nabla_{dim}^{2} (\psi^{f} - \psi^{o})_{k} = \frac{2\Omega}{a f^{2} \sqrt{\bar{F}_{k}}} \sum_{j} e_{k_{j}} \nabla^{2} \lambda_{j} .$$

(∇^2 is dimensional on the left side, nondimensional on the right side.)

Using (4.3), (4.6), and (4.8), we obtain

$$\frac{1}{f^{2}} \nabla_{dim}^{2} (\phi^{f} - \phi^{o})_{k} = \left(\frac{\bar{F}_{\ell}}{\bar{F}_{k}}\right)^{1/2} \left(\frac{\nabla \cdot \underset{\sim}{A}^{o} \ell}{f^{2}}\right)_{center} \sum_{j} e_{k_{j}} e_{\ell_{j}} (\mathcal{U}_{j}, W_{j}),$$
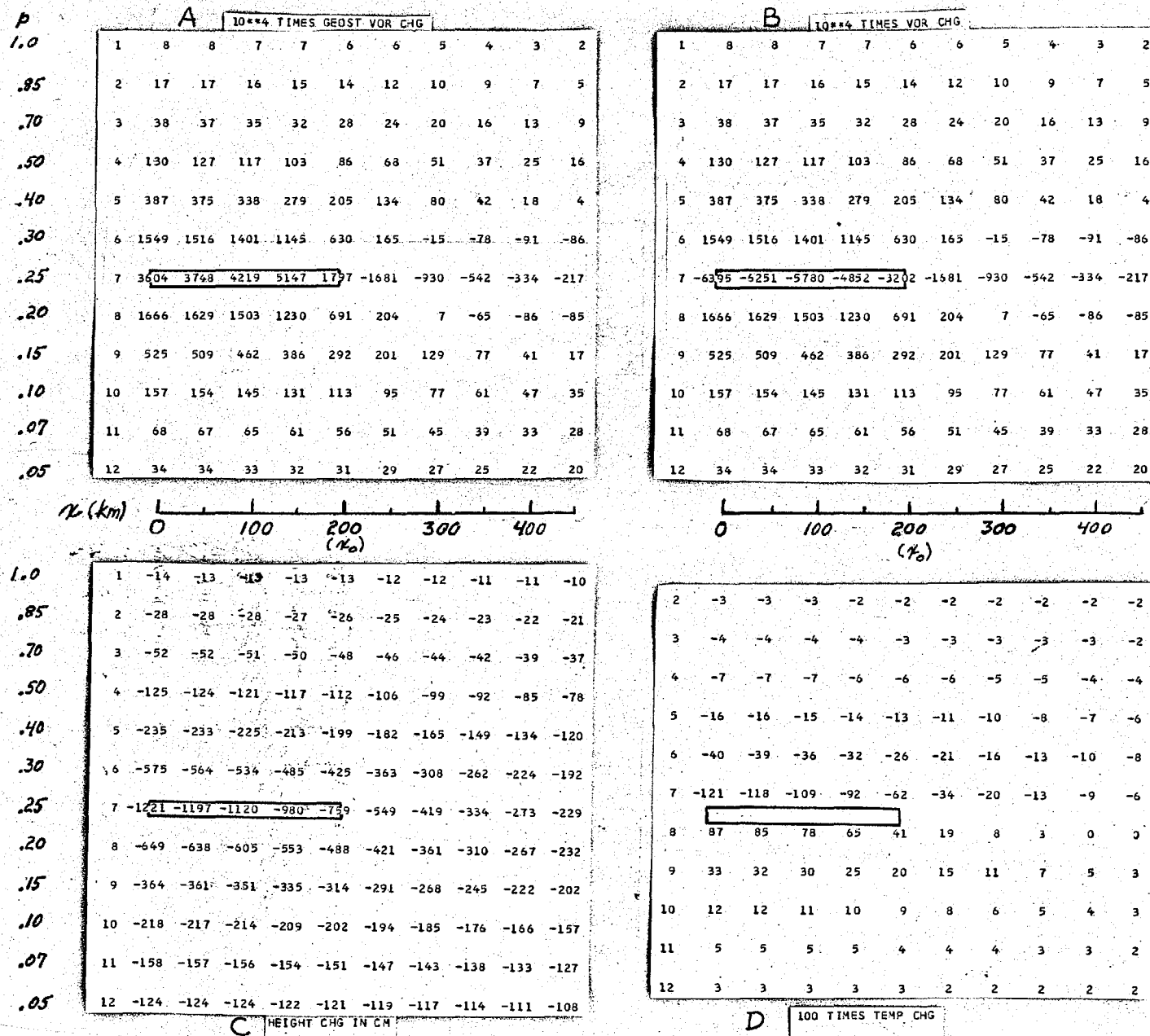
$$\frac{1}{f} \nabla_{dim}^{2} (\psi^{f} - \psi^{o})_{k} = \left(\frac{\bar{F}_{\ell}}{\bar{F}_{k}}\right)^{1/2} \left(\frac{\nabla \cdot \underset{\sim}{A}^{o} \ell}{f^{2}}\right)_{center} \sum_{j} e_{k_{j}} e_{\ell_{j}} (1 + \mathcal{U}_{j}, W_{j})$$

(4.9)

in which the first term in the parenthetical expression is used for $\mathcal{N} < \mathcal{N}_{o}$,

and the second for $\mathcal{N} > \mathcal{N}_{o}$. The subscript "center" on $\nabla \cdot \underset{\sim}{A}^{o}$ is a

reminder that it is the constant value assumed for $\mathcal{N} < \mathcal{N}_{o}$ at level ℓ.

These expressions are most easily interpreted if we fix attention on

a particular case--a small isolated "cyclonic" region ($\mathcal{N} < \mathcal{N}_{o}$) in which

$\nabla \cdot \underset{\sim}{A}^{o}_{\ell}$ is negative. From (1.2) we can imagine this as resulting from

$\nabla^{2} \phi_{\ell}^{o}$ not being large enough (algebraically) or from $\nabla^{2} \psi_{\ell}^{o}$ being too

large (algebraically) to balance the nonlinear ψ^{o} terms in (1.2).

Figure 1. Distribution of vorticity, height and temperature changes resulting from $\nabla \cdot \mathbf{A}^\circ = -f^2$ at level 7 (250 mb) in the circular region $0 < r < 100\ km$ (indicated by the open bars).

A — 104 TIMES GEOST VOR CHG**

p										
1.0	1	8	8	7	7	6	6	5	4	3	2
.85	2	17	17	16	15	14	12	10	9	7	5
.70	3	38	37	35	32	28	24	20	16	13	9
.50	4	130	127	117	103	86	68	51	37	25	16
.40	5	387	375	338	279	205	134	80	42	18	4
.30	6	1549	1516	1401	1145	630	165	-15	-78	-91	-86
.25	7	3604	3748	4219	5147	1797	-1681	-930	-542	-334	-217
.20	8	1666	1629	1503	1230	691	204	7	-65	-86	-85
.15	9	525	509	462	386	292	201	129	77	41	17
.10	10	157	154	145	131	113	95	77	61	47	35
.07	11	68	67	65	61	56	51	45	39	33	28
.05	12	34	34	33	32	31	29	27	25	22	20

x (km) scale: 0   100   200   300   400 (r_0)

B — 104 TIMES VOR CHG**

p										
1.0	1	8	8	7	7	6	6	5	4	3	2
.85	2	17	17	16	15	14	12	10	9	7	5
.70	3	38	37	35	32	28	24	20	16	13	9
.50	4	130	127	117	103	86	68	51	37	25	16
.40	5	387	375	338	279	205	134	80	42	18	4
.30	6	1549	1516	1401	1145	630	165	-15	-78	-91	-86
.25	7	-6395	-5251	-5780	-4852	-3202	-1681	-930	-542	-334	-217
.20	8	1666	1629	1503	1230	691	204	7	-65	-86	-85
.15	9	525	509	462	386	292	201	129	77	41	17
.10	10	157	154	145	131	113	95	77	61	47	35
.07	11	68	67	65	61	56	51	45	39	33	28
.05	12	34	34	33	32	31	29	27	25	22	20

scale: 0   100   200   300   400 (r_0)

C — HEIGHT CHG IN CM

p										
1.0	1	-14	-13	-13	-13	-13	-12	-12	-11	-11	-10
.85	2	-28	-28	-28	-27	-26	-25	-24	-23	-22	-21
.70	3	-52	-52	-51	-50	-48	-46	-44	-42	-39	-37
.50	4	-125	-124	-121	-117	-112	-106	-99	-92	-85	-78
.40	5	-235	-233	-225	-213	-199	-182	-165	-149	-134	-120
.30	6	-575	-564	-534	-485	-425	-363	-308	-262	-224	-192
.25	7	-1221	-1197	-1120	-980	-739	-549	-419	-334	-273	-229
.20	8	-649	-638	-605	-553	-488	-421	-361	-310	-267	-232
.15	9	-364	-361	-351	-335	-314	-291	-268	-245	-222	-202
.10	10	-218	-217	-214	-209	-202	-194	-185	-176	-166	-157
.07	11	-158	-157	-156	-154	-151	-147	-143	-138	-133	-127
.05	12	-124	-124	-124	-122	-121	-119	-117	-114	-111	-108

D — 100 TIMES TEMP CHG

p										
.85	2	-3	-3	-3	-2	-2	-2	-2	-2	-2	-2
.70	3	-4	-4	-4	-4	-3	-3	-3	-3	-3	-2
.50	4	-7	-7	-7	-6	-6	-6	-5	-5	-4	-4
.40	5	-16	-16	-15	-14	-13	-11	-10	-8	-7	-6
.30	6	-40	-39	-36	-32	-26	-21	-16	-13	-10	-8
.25	7	-121	-118	-109	-92	-62	-34	-20	-13	-9	-6
.20	8	87	85	78	65	41	19	8	3	0	0
.15	9	33	32	30	25	20	15	11	7	5	3
.10	10	12	12	11	10	9	8	6	5	4	3
.07	11	5	5	5	5	4	4	4	3	3	2
.05	12	3	3	3	3	3	2	2	2	2	2

Considering first the outer region $\mathcal{N} > \mathcal{N}_o$, we see from (4.9) that the two vorticity changes are equal. In this way the presumed satisfaction of (1.2) at $\mathcal{N} > \mathcal{N}_o$ is maintained. At the forcing level itself (k = ℓ), the fact that W_j and $e_{k_j} e_{\ell_j}$ are both positive shows that $\nabla^2 \phi$ and $\nabla^2 \psi$ are both decreased in this outer region. In other words, the original "deficit" of $\nabla^2 \phi$ in the central region at level ℓ is dispersed laterally outward, but without destroying the balance present originally in the outer region.

At the interface $\mathcal{N} = \mathcal{N}_o$, $\nabla^2 \psi'$ is continuous, but $\nabla^2 \phi'$ has a discontinuity. In the central region, again at level k = ℓ, $\nabla^2 \phi_\ell$ is increased in this example, since both $\nabla \cdot \underset{\sim}{A}^o$ and $\mathcal{U}_j$ are negative.

$\nabla^2 \psi_\ell$ is decreased ($1 + \mathcal{U}_j$ being positive). Thus $\nabla^2 \phi'_\ell$ and $\nabla^2 \psi'_\ell$ both participate in correcting the original imbalance in the central region at level ℓ. In the central region at levels k different from ℓ, the sum ($\nabla^2 \phi'_k - f \nabla^2 \psi'_k$) is zero because $\sum_j e_{k_j} e_{\ell_j}$ vanishes for k ≠ ℓ.

In a similar way, we can derive the relation

$$ (\phi^f - \phi^o)_k = f^2 a^2 \left(\frac{\bar{\tau}_\ell}{\bar{\tau}_k} \right)^{1/2} \left(\frac{\nabla \cdot \underset{\sim}{A}^o}{f^2} \ell \right)_{center} \sum_j \frac{e_{k_j} e_{\ell_j}}{\sigma_j^2} (1 + \mathcal{U}_j, W_j) \tag{4.10} $$

and also use this to evaluate $T'_k = \mathcal{U}_k (\phi'_k - \phi'_{k-1})$ at $k = 2, \cdots, K.$

These functions have been computed using the same distribution of $\mathcal{N}_k, \bar{z}, \bar{\beta}_k$ and $\bar{\tau}_k$ used for Table I, and with sin Θ = 0.707, $\mathcal{N}_o = (200 km / a)$ and ($\nabla \cdot \underset{\sim}{A}^o_\ell / f^2$) = -1. The results for ℓ = 7 (250 mbs) are shown in figure 1. Parts (A) and (B) of figure 1 verify

the deductions discussed above. (The printed values for $\nabla^2\phi'/f^2$ at $\varkappa = \varkappa_o$ in part (A) are the averages of the inner and outer solutions at $\varkappa = \varkappa_o$.) The lateral dispersion of negative $\nabla^2\phi$ at $\ell = 7$ from the inner to the outer region is clear. As a counterpart to this, we see that $\nabla^2\psi'$ in the central region at $\ell = 7$ is negative, and the original "excess" of $\nabla^2\psi^o$ at $\varkappa < \varkappa_o$, $\ell = 7$, is dispersed <u>vertically</u> in this region, since $\nabla^2\psi'$ is positive at $k \neq 7$ for $\varkappa < \varkappa_o$.

The distribution of height change $3' = \phi'/g$ in part (C) of figure 1 is readily understood from the distribution of $\nabla^2\phi'_o$. It corresponds to a fall in height centered at $k = 7$, $x = 0$. This leads of course to the temperature decreases shown in part (D) at $k \leq \ell$ and temperature increases at $k > \ell$. (Recall that $T_k = \tau_k(\phi_k - \phi_{k-1})$, so that $T'_7 = \tau_7(\phi'_7 - \phi'_o)$.)

The values of $(\delta\phi_1)^2 = (26 \text{ m}^2\text{sec}^{-2})^2$, $(\delta T)^2 = (0.6 \text{ deg})^2$ and $(\delta v)^2 = (2.5 \text{ m/sec})^2$ have in this case produced a ratio of geostrophic vorticity change to vorticity change of

$$\Gamma = \left[\frac{1}{f}\nabla^2\phi'/\nabla^2\psi'\right] = \left(\frac{3604}{6395}\right)^2 = 0.318 \qquad (4.11)$$

at $x = 0$, $k = \ell = 7$. In doing so we have minimized the area integral of

$$\sum_{k=1} \bar{\tau}_k (\nabla\psi'_k)^2 + \sum_{k=2} \bar{\beta}_k \tau_k^2 (\phi'_k - \phi'_{k-1})^2 + \bar{\alpha}(\phi'_1)^2$$

We can therefore expect Γ to increase if $\bar{\tau}$ is increased (δv decreased) and if $\bar{\beta}$ and $\bar{\alpha}$ are decreased (δT and $\delta\phi_1$ increased).

This dependence is complicated. In order to provide some numerical insight into it, a series of computations have been made in which the ratio $\delta\phi_1/\delta T$ is kept fixed at $(26/.6)$ $m^2sec^{-2}deg^{-1}$, the uniformity of δT and δv with k is retained, but both $\sin\theta$ and the ratio $(\delta T/\delta v)$ have been varied. The results can be condensed into an approximate formula:

$$\Gamma \approx \left[\frac{6.63 \cdot R \, \delta T}{2\Omega a \, \sin\theta \, \delta v} \right]^{3.11}$$

(4.12)

(N_0 has been kept fixed at 200 km in this procedure.) $\nabla^2\phi'$ and $\nabla^2\psi'$ will therefore participate equally in restoring the balance equation at $x = 0$, $p = .25$ in this special case, when

$$\frac{\delta T(deg)}{\delta v(m \ sec^{-1})} = 0.48786 \ \sin\theta$$

(4.13)

Although determined for a special case, this case of localized imbalance at $p = .25$ is realistic enough that (4.12) and (4.13) may be useful in adjusting the effects of different δT and δv.

5. Effect of a horizontal variation in analysis weights

The functions γ' and β' represent the horizontal variation of $\overline{(\delta w^2)}/\delta w^2$ and $\overline{(\delta T^2)}/\delta T^2$, with a mean value of one. They will therefore be less than one over "oceanic" regions and greater than one over the data-rich continents and, to a considerable extent, will be quite similar to one another: $\gamma' \sim \beta'$. Using Cartesian coordinates x,y, we can examine most simply the effect of this horizontal variation by considering the step function distribution

$$\beta' = \gamma' = \begin{array}{l} \gamma_I : (x > 0) \quad \text{(Region I)} \\[2mm] \gamma_{II} : (x < 0) \quad \text{(Region II)} \end{array} \qquad (5.1)$$

This is introduced into equations (4.1) together with a forcing function

$$F_j = G_j \sin my \, \sin(px + \nu). \qquad (5.2)$$

γ_I, γ_{II} and G_j are constants. ν is a phase angle, and m and p are positive wave numbers.

The equations in both regions are the same:

$$h_j = G_j \, u(x) \sin my, \qquad (5.3)$$

$$s_j = G_j \, v(x) \sin my,$$

and

$$\left(\frac{d^2}{dx^2} - m^2\right)(u - v) = -\sin(px + \nu), \qquad (5.4)$$

$$\left(\frac{d^2}{dx^2} - m^2\right)v - \sigma_j^2 u = 0.$$

The internal boundary conditions at x = 0 are continuity in h, s, $\partial h/\partial x$, and $\partial(\gamma's)/\partial x$.

These translate into

$$
\begin{pmatrix} u \\ v \\ \dfrac{du}{dx} \\ \gamma_{II} \dfrac{dv}{dx} \end{pmatrix}_{x=-\epsilon} = \begin{pmatrix} u \\ v \\ \dfrac{du}{dx} \\ \gamma_{I} \dfrac{dv}{dx} \end{pmatrix}_{x=+\epsilon}
\tag{5.5a}
$$

as $\epsilon \to 0$. Also, the solutions must be bounded at infinity:

$$
\lim_{|x| \to \infty} (u, v) = \text{finite}. \tag{5.5b}
$$

The solution for u is simply the particular solution

$$
u = \frac{\sin(px + \nu)}{p^2 + m^2 + \sigma_j^2} \quad , \quad -\infty \leq x \leq +\infty , \tag{5.6a}
$$

since the boundary conditions (5.5a) and (5.5b) rule out the homogeneous solution $\exp \pm(m^2 + \sigma_j^2)x$ for u. The solution for v is

$$
v = \frac{-\sigma_j^2}{(m^2 + p^2)(m^2 + p^2 + \sigma_j^2)} \left[\sin(px+\nu) + \left(\frac{\gamma_I - \gamma_{II}}{\gamma_I + \gamma_{II}} \right) \frac{p \cos \nu}{m} e^{\pm mx} \right], \tag{5.6b}
$$

with the + sign taken in region II (x < 0) and the − sign taken in region I (x > 0). (We consider m > 0.)

The $\sin(p_x + \nu)$ part of u and v is the solution for any completely
uniform value of γ'. The introduction of the discontinuity (5.1) into
γ and β' therefore does not change the geopotential field h_j. √
But it does change the "streamfunction" field by an amount

$$\psi_j' = \frac{G_j \, \sigma_j^2}{(m^2 + p^2)(m^2 + p^2 + \sigma_j^2)} \left(\frac{\gamma_{II} - \gamma_I}{\gamma_{II} + \gamma_I}\right) \frac{p \cos \nu}{m} \sin m y \, e^{\pm m x} \tag{5.7}$$

This produces a discontinuity in the y-component of velocity across the
line x = 0, i.e., a vortex sheet. The discontinuity vanishes when the
phase ν equals $\pm \pi/2$, i.e., when the forcing function $\sin(p_x + \nu)$
has its maximum or minimum at x = 0. The discontinuity in γ' does not
change the vorticity field away from the line x = 0, since $\nabla^2 \psi_j' = 0$
for x ≠ 0. The strength of this vortex sheet is measured by the discontinuity
in meridional velocity

$$\Delta V_k \, (\text{dimen}) = \lim_{\epsilon \to 0} \frac{1}{a} \left[\left(\frac{\partial \psi_k'}{\partial x}\right)_{x = +\epsilon} - \left(\frac{\partial \psi_k'}{\partial x}\right)_{x = -\epsilon} \right] \tag{5.8}$$

√ h_j does not change in this case only because $\nabla^2 \psi_j' = 0$. For more
general variations in γ', h_j will differ from its value for $\gamma' \equiv 1$.

Equation (5.2) is consistent with the following spatial distribution

of $\nabla \cdot \underset{\sim}{A}^o$:

$$\left(\frac{\nabla \cdot \underset{\sim}{A}^o}{f^2}\right)_\ell = Q_\ell \sin my \sin(px+\nu)$$

(5.9)

For simplicity we set Q_ℓ equal to a linear increase from zero at p = 1

to Q_{max} at p = .25 and a linear decrease from Q_{max} at p = .25 to zero

again at p = 0. G_j in (5.7) then equals $(f^2 a / 2\Omega) \sum_\ell e_{\ell_j} \sqrt{\bar{\tau}_\ell} \, Q_\ell$,

and (5.8) can be shown to equal

$$\Delta V_k = \frac{2fa}{(m^2+p^2)} \, p \cos\nu \, \sin my \left(\frac{\tau_I - \tau_{II}}{\tau_I + \tau_{II}}\right) \sum_\ell Q_\ell \left(\frac{\bar{\tau}_\ell}{\bar{\tau}_k}\right)^{\frac{1}{2}} \sum_j \frac{e_{k_j} \cdot e_{\ell_j} \, \sigma_j^2}{m^2+p^2+\sigma_j^2} \; .$$

A value of $(0.25)^2$ for Q_{max} represents a moderately intense wave system.

For definiteness we locate x = 0 at the west coast of a continent and,

in agreement with the discussion of τ' and β' in section 3, we choose

$\tau_I = 2$, $\tau_{II} = 0.2$. By selecting $\nu = 0$ we position the wave in

$\nabla \cdot \underset{\sim}{A}^o$ to correspond to a "cyclonic" region of negative $\nabla \cdot \underset{\sim}{A}^o$

centered one-quarter wavelength off the coast. For the nondimensional

wave numbers p and q, we assign reasonable values of

$$p = \frac{2\pi a}{\text{east-west wavelength}} = 4\pi$$

$$m = \frac{2\pi a}{\text{north-south wavelength}} = 2\pi$$

corresponding to values of a/2 for the east-west wavelength and a for the

north-south wavelength.

Values of ΔV for the above parameter values, for $\sin\theta = 0.707$, and the same $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ used for Table I, are shown in Table II (at $\sin(m\, y) = 1$). The discontinuity ΔV is equal to $2(\gamma_I - \gamma_{II})/(\gamma_I + \gamma_{II})$ times the continuous meridional velocity V at x = 0 due to the $\sin(p\psi + \nu)$ term in (5.6b). It is therefore not large enough to change the sign of the meridional velocity from the left side to the right side of the line x = 0 in this example. This suggests that it will be practical to use horizontal distributions of γ' and β' in (3.6) or (3.11) which change suddenly--for example in oceanic areas where an overall level of $\beta' \sim \gamma' \sim 0.2$ is punctuated by small regions of $\beta' \sim \gamma' \sim 2$ centered on scattered island and ship stations.

The predominant positive sign of ΔV in Table II can be deduced from equation (3.11) and the results of section 4. In the given area, $\nabla \cdot \underline{A}^o$ is zero at x = 0 and negative to the west (x < 0). For uniform γ' and β' we therefore expect a center of positive streamfunction change west of x = 0 and a negative center to the east of x = 0, giving a negative value of $\partial S/\partial \psi$ at x = 0. Referring to equation (3.11b),

$$\nabla \cdot \gamma' \nabla S_j + \nabla \cdot \sin\theta \, \nabla R_j = 0, \tag{3.11b}$$

we can imagine the introduction of a non-uniform γ' as resulting in an approximate equation:

$$\gamma' \nabla^2 S_j \text{ (new)} + \nabla \cdot \sin\theta \, \nabla R_j \approx -\nabla\gamma' \cdot \nabla S_j \text{ (old)}$$

The right side of this equation would be positive at x = 0 for the case shown in Table II.

Table II. Values of ΔV at $m y = \pi/2$. This is the discontinuity in meridional velocity produced at a west coast ($x = 0$) for a realistic forcing function $\nabla \cdot \underset{\sim}{A}^{o} = f^2 Q_{\ell} \sin my \sin p\#$. The continuous meridional velocity V at $x = 0$ is also tabulated.

p	Q	ΔV (m sec^{-1})	V (m sec^{-1})
1.00	0.0000	− 0.272	− .166
0.85	0.0125	0.304	0.186
0.70	0.0250	0.786	0.480
0.50	0.0417	1.253	0.766
0.40	0.0500	1.435	0.877
0.30	0.0583	1.617	0.988
0.25	0.0625	1.766	1.079
0.20	0.0500	0.897	0.548
0.15	0.0375	0.181	0.111
0.10	0.0250	− 0.332	− 0.203
0.07	0.0175	− 0.528	− 0.323
0.05	0.0125	− 0.608	− 0.372

The following argument illustrates the minimization principle acting here. In the figure, let ψ'_{max} and ψ'_{min} denote the circulation change pattern for $\gamma' = \beta' = 1$. The superposition of (5.7) from non-uniform γ acts to reduce $(\nabla\psi')^2$ over the continental area (where γ' is large) and increase it over the oceanic area (where γ' is small).

ocean

small γ'

continent

large γ'
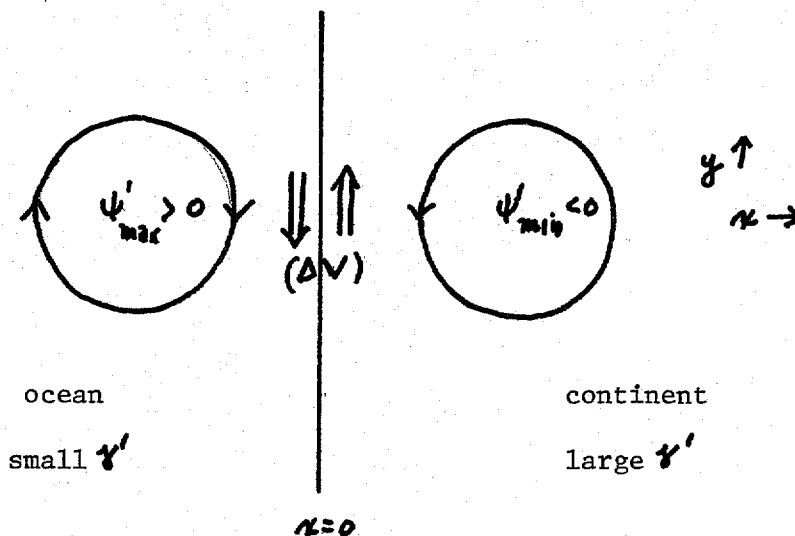
6. Iterative solution methods

The problem is to get solutions for H_j, S_j and R_j in equations (3.11) in which γ', β', F_j are known functions and λ_j is a known positive constant. In this section we first consider some simple iterative techniques for the simple Cartesian case of $\gamma' = \beta' = 1$ and $\sin\theta$ = constant, i.e., equations (4.2) with $\gamma' = \beta' = 1$. We shall see in this section that conventional iteration methods will probably not be satisfactory, even for this simplified problem. (Stephens also experienced difficulty in this respect; see page 737 of his 1970 paper in J. Applied Meteor.).

In order to explore iterative methods most expeditiously, we shall consider that "simultaneous relaxation" is performed. That is to say, if we have an equation

$$\nabla^2 a^{\nu+1} = b^\nu$$

for the $\nu+1$ iteration of variable a given a known value of the ν iteration of variable b, we obtain $a^{\nu+1}$ exactly before proceeding to the next iteration. (It seems reasonable that if conventional point-by-point relaxation schemes were used on a horizontal grid, they would converge even slower than this hypothetical simultaneous procedure.) This enables us to replace the operator ∇^2 by a simple factor, $-n^2$, (n is roughly equivalent to the spherical harmonic wave number). We of course now have to require convergence for __all__ n for each σ_j.

Equations (4.2) then reduce to

$$-n^2 h_j + n^2 q_j = -F_j ,$$

$$q_j + r_j = 0 , \qquad (6.1)$$

$$-n^2 r_j + \sigma_j^2 h_j = 0 .$$

The solution is

$$h_j = F_j / (n^2 + \sigma_j^2) ,$$

$$q_j = -\sigma_j^2 F_j / n^2 (n^2 + \sigma_j^2) , \qquad (6.2)$$

$$r_j = \sigma_j^2 F_j / n^2 (n^2 + \sigma_j^2) .$$

This solution is of course not accessible to us in solving (4.2), but we

can simplify our algebra here by working instead with the "error"

$$(h, s, r)_{(error)} = (h_j, s_j, r_j)_{approx} - (h_j, s_j, r_j)_{true}$$

These lead to the homogeneous error equations

$$- h + s = 0, \tag{6.3a}$$

$$s + r = 0, \tag{6.3b}$$

$$- n^2 r + \sigma^2 h = 0. \tag{6.3c}$$

Iteration scheme I.

$$h^{\nu+1} = s^{\nu},$$

$$n^2 r^{\nu+1} = \sigma^2 h^{\nu+1}, \tag{6.4a}$$

$$s^{\nu+1} = - r^{\nu+1}.$$

These lead to the convergence process

$$h^{\nu+1} = - \frac{\sigma^2}{n^2} h^{\nu} \tag{6.4b}$$

for h, r, and s. Convergence is thereby achieved for short waves

$$n^2 > \lambda_j \sin^2 \theta \tag{6.4c}$$

Since λ_j varies from 100 to 32000, this scheme is unsatisfactory by itself.

By reversing the equations, we obtain

Iteration scheme II

$$s^{\nu+1} = h^{\nu},$$
$$r^{\nu+1} = s^{\nu+1},$$
$$\sigma^2 h^{\nu+1} = n^2 r^{\nu+1},$$

(6.5a)

which understandably gives the reverse of (6.4b):

$$h^{\nu+1} = -\frac{n^2}{\sigma^2} h^{\nu}.$$

(6.5b)

This converges for <u>long waves</u>

$$n^2 < \lambda_j \sin^2\theta$$

(6.5c)

Neither this nor (6.4c) is satisfactory for all n.

Iteration scheme III

This is a mixture of I and II:

$$h^* = s^{\nu}, \quad n^2 r^* = \sigma^2 h^*, \quad s^* = -r^*,$$
$$s^{**} = h^{\nu}, \quad r^{**} = -s^{**}, \quad \sigma^2 h^{**} = n^2 r^{**},$$

(6.6a)

followed by the combination

$$h^{\nu+1} = \alpha h^* + (1-\alpha)h^{**},$$
$$s^{\nu+1} = \beta s^* + (1-\beta)s^{**},$$

(6.6b)

where α and β are numbers between zero and 1. These produce the iteration results

$$h^{\nu+1} = -\frac{n^2(1-\alpha)}{\sigma^2} h^{\nu} + \alpha s^{\nu},$$
$$s^{\nu+1} = (1-\beta)h^{\nu} - \frac{\sigma^2}{n^2} \beta s^{\nu}.$$

(6.6c)

The iteration eigenvalue μ for this process satisfies the quadratic

$$\mu^2 + \left[(1-\alpha)\frac{n^2}{\sigma^2} + \beta\frac{n^2}{\sigma^2}\right]\mu + (\beta-\alpha) = 0,$$

with solutions

$$2\mu = (1-\alpha)N + \frac{\beta}{N} \pm \left\{\left[(1-\alpha)N+\frac{\beta}{N}\right]^2 + 4\alpha-4\beta\right\}^{1/2} \tag{6.6d}$$

N stands here for n^2/σ^2, a quantity which effectively varies from almost zero to a very large number.

For fixed α and β the radicand has a minimum with respect to N at $N^2 = \beta/(1-\alpha)$, this minimum value being $4\alpha(1-\beta)$. The square root is therefore real. Since $(1-\alpha)N + \beta/N$ is positive, the convergence criterion is

$$\mu_{max} = \frac{1}{2}\left\{(1-\alpha)N + \frac{\beta}{N} + \sqrt{\left[(1-\alpha)N +\frac{\beta}{N}\right]^2 + 4(\alpha-\beta)}\right\} < 1$$

for all N. For small N,

$$\mu_{max} = \frac{\beta}{N} + \frac{N\alpha(1-\beta)}{\beta} + O(N^3)$$

For large N,

$$\mu_{max} = (1-\alpha)N + \frac{\alpha(1-\beta)}{(1-\alpha)N} + O(N^{-3}).$$

The requirement that β be small and that $(1-\alpha)$ be small suggests that we choose $\beta = 1-\alpha$. This gives the simpler condition

$$\mu_{max} = (1-\alpha)Z + \sqrt{(1-\alpha)^2 Z^2 + 2\alpha - 1} \tag{6.6e}$$

where

$$Z = \frac{1}{2}\left(N + \frac{1}{N}\right) > 1,$$

$$N = n^2/\lambda_j \cdot \sin^2\theta$$

The large range in $m^2/\sin^2\theta$ means that Z can be very large, and the only hope for satisfying (6.6e) is to have α close to 1. However, μ_{max} equals one for $\alpha=1$ and $\partial\mu_{max}/\partial\alpha$ at $\alpha=1$ equals 1-Z, which is less than zero. Therefore μ_{max} will be greater than one for α slightly less than one. This scheme will not work.

An only slightly more successful scheme is the following.

Iteration scheme IV

$$h^{\nu} = s^{\nu}$$

$$r* = (\sigma^2/n^2)h^{\nu}$$

$$s* = - r^{\nu}$$

$$r** = - s^{\nu}$$

$$h* = (n^2/\sigma^2)r^{\nu}$$

$$s** = h* = (n^2/\sigma^2)r^{\nu}$$

followed by

$$\ell^{\nu+1} = \alpha \ell^* + (1-\alpha)\ell^{**}$$

$$\hbar^{\nu+1} = \alpha\hbar^* + (1-\alpha)\hbar^{**}$$

This leads to the iteration scheme

$$\ell^{\nu+1} = - \left\{ [\alpha - (1-\alpha)N]^2 /N \right\} \ell^{\nu-1}$$

where N again equals n^2/σ^2. The convergence criterion is

$$[\alpha(1+N)-N]^2 < N$$

For a fixed α, N must lie between

$$N_1(\alpha) = \frac{1 + 2\alpha - 2\alpha^2 - \sqrt{1 + 4\alpha - 4\alpha^2}}{2(1-\alpha)^2}$$

and

$$N_2(\alpha) = \frac{1 + 2\alpha - 2\alpha^2 + \sqrt{1 + 4\alpha - 4\alpha^2}}{2(1-\alpha)^2}$$

in order to have convergence. The values of N_1 and N_2 are as follows, ε being a small quantity:

		N_1	N_2
a)	$\alpha = \epsilon$	ϵ^2	$1 + 4\epsilon \ \cdots$
b)	$\alpha = \frac{1}{2}$	$3 - 2\sqrt{2} \approx 0.17$	$3 + 2\sqrt{2} \approx 5.83$
c)	$\alpha = 1 - \epsilon$	$1 - 4\epsilon$	$\frac{1}{\epsilon^2} + \frac{2}{\epsilon} - 3 + 4\epsilon \ \cdots$

N is equal to $n^2(\lambda_j \cdot \sin^2\theta)^{-1} \approx 2n^2/\lambda_j$

For $j = 1$ and 12 we have then, with $n_{min} = 1$, the following ranges for N:

$$j = 1 \qquad .02 \leq N < \infty$$

$$j = 12 \qquad .000067 \leq N \leq \infty$$

Both $j = 1$ and $j = 12$ will have their lower limit N_1 satisfied only by small α (α = ε) but the upper limit N_2 will not be satisfied. This method is therefore unsatisfactory also.

An iterative method to solve even the simplified system (6.1) does not seem possible. We therefore turn to a more direct method.

7. Use of spherical harmonics

We consider now the complete system (3.11), dropping the j subscripts for simplicity:

$$\nabla^2 H - \nabla \cdot \sin\theta \, \nabla S = -F,$$

$$\nabla \cdot \gamma' \nabla S + \nabla \cdot \sin\theta \, \nabla R = 0,$$

$$\nabla^2 R + \lambda \beta' H = 0.$$

We first rewrite this to put all variable γ' and β' effects on the right side:

$$\nabla^2 H - \nabla \cdot \sin\theta \, \nabla S = -F, \tag{7.1}$$

$$\nabla^2 S + \nabla \cdot \sin\theta \, \nabla R = \nabla \cdot (1-\gamma') \nabla S, \tag{7.2}$$

$$\nabla^2 R + \lambda H = \lambda (1-\beta') H. \tag{7.3}$$

The iteration process will consist of using the previous iterates S^ν and H^ν (initially zero) for the right sides of (7.2) and (7.3), and determining $H^{\nu+1}$, $S^{\nu+1}$ and $R^{\nu+1}$ from the left side of (7.1)-(7.3). [The process for the special case $\gamma' = \beta' = 1$ explored in section 6 would not be iterative, and therefore not subject to those convergence problems.]

We use the normalized spherical harmonics to expand these fields, for example

$$Y_n^m = P_n^m e^{im\lambda},$$

$$H = \sum_{m,n} H_n^m Y_n^m \tag{7.4}$$

This is done for the three unknowns $H^{\nu+1}, S^{\nu+1}, R^{\nu+1}$ on the left sides of (7.1)-(7.3). (S_0^0 and R_0^0 are defined to be zero.) Let us also consider that the right sides of (7.1)-(7.3) have also been expanded in spherical harmonics, perhaps by a transform technique, for a given ν iteration.

The operator ∇^2 on (7.4) produces the well-known result

$$\nabla^2 H = - \sum n(n+1) H_n^m Y_n^m .$$

The operator $\nabla \cdot \sin\theta \, \nabla S$ however becomes

$$\nabla \cdot \sin\theta \, \nabla S = \sin\theta \, \nabla^2 S + \cos\theta \, \partial S/\partial\theta$$

$$= x \nabla^2 S + (1-x^2) \partial S/\partial x$$

so that

$$\nabla \cdot \sin\theta \, \nabla S = \sum_n S_n^m \left[-n(n+1)x \, Y_n^m + (1-x^2) \partial Y_n^m/\partial x \right]$$

(We have introduced the conventional notation $x = \sin\theta$ at this point.) Using the relations (8.5.3), (8.5.4) and (8.14.13) in Abramowitz and Stegun, we can show that

$$-n(n+1)x \, Y_n^m + (1-x^2)\frac{\partial Y_n^m}{\partial x} = -n(n+2) a_{n+1}^m Y_{n+1}^m - (n^2-1) a_n^m Y_{n-1}^m \tag{7.5}$$

in which

$$a_n^m = \left[\frac{n^2 - m^2}{4n^2 - 1} \right]^{1/2} , \quad a_n^n = 0 . \tag{7.6}$$

(Note that a_0^0 is not needed, since S_0^0 and R_0^0 are zero.)

For simplicity let us now consider only a single zonal wave number m and temporarily suppress the m superscripts and subscripts. Equations (7.1)-(7.3) are

$$-\sum_{m}^{N_m} n(n+1) H_n P_n + \sum_{n}^{N_m} S_n \left[n(n+2) a_{n+1} P_{n+1} + (n^2-1) a_n P_{n-1} \right] = -\sum_{n}^{N_m} F_n P_n ,$$

$$-\sum_{n}^{N_m} n(n+1) S_n P_n - \sum_{n}^{N_m} R_n \left[n(n+2) a_{n+1} P_{n+1} + (n^2-1) a_n P_{n-1} \right] = \sum_{n}^{N_m} A_n P_n$$

$$\sum_{n}^{N_m} \left[\lambda H_n - n(n+1) R_n \right] P_n = \sum_{n}^{N_m} B_n P_n$$

where F_n, A_n, and B_n denote the expansion coefficients of the right sides of (7.1)-(7.3). Collecting corresponding coefficients now gives us three equations for n = m, m+1, ..., N_m-1, N_m:

$$-n(n+1) H_n + (n^2-1) a_n S_{n-1} + n(n+2) a_{n+1} S_{n+1} = -F_n , \tag{7.7a}$$

$$n(n+1) S_n + (n^2-1) a_n R_{n-1} + n(n+2) a_{n+1} R_{n+1} = -A_n , \tag{7.7b}$$

$$\lambda H_n - n(n+1) R_n = B_n . \tag{7.7c}$$

The a_n terms in (7.7a) and (7.7b) disappear for n = m. At n = N_m, the S_{n+1} and R_{n+1} terms in (7.7a) and (7.7b) vanish. (We can simply define $a_n^m \equiv 0$ for $n > N_m$.) Elimination of H_m reduces this to

$$(m^2-1)a_m S_{m-1} + m(m+2)a_{m+1} S_{m+1} - m^2(m+1)^2 R_m / \lambda = -F_m + m(m+1)B_m / \lambda \quad (7.8a)$$

$$m(m+1)S_m + (m^2-1)a_m R_{m-1} + m(m+2)a_{m+1} R_{m+1} = -A_m \quad (7.8b)$$

Elimination of S_n is now convenient. If we define

$$b_m = \frac{(m-1)(m+1)}{m} a_m = \frac{m^2-1}{m} \sqrt{\frac{m^2-m^2}{4m^2-1}}, \quad (7.9)$$

the result is an equation for R_n:

$$b_m b_{m-1} R_{m-2} + \left[\frac{m b_m^2}{m-1} + \frac{(m+1)b_{m+1}^2}{m+2} + \frac{m^2(m+1)^2}{\lambda} \right] R_m + b_{m+1} b_{m+2} R_{m+2}$$

$$= F_m - \frac{m(m+1)}{\lambda} B_m - \left[\frac{b_m}{m-1} A_{m-1} + \frac{b_{m+1}}{m+2} A_{m+1} \right] \quad (7.10)$$

(Note that $b_m^2/(m-1)$ vanishes for $n = 1$ and that b_n vanishes for $n = m$ and for $n > N_m$). The matrix multiplying the column vector R_n ($n = m$, $m+1$, ..., N_m) is symmetric, with entries only along the principal diagonal and the upper and lower diagonals twice removed from the main diagonal. (The latter feature reflects the fundamental separation of P_n^m into even and odd polynomials.)

Having solved for R_n from (7.10), we find S_n from (7.8b) and H_n from (7.7c). These are the new iterates $H^{\nu+1}$ and $S^{\nu+1}$. The right sides of (7.2) and (7.3) can then be reevaluated with the new H and S, and the process repeated.

An interesting question is the truncation limits to be assigned to m and N_m in the expansions (7.4). These limits should be large enough to

include the information content in the original forcing field $\nabla \cdot \underset{\sim}{A}^{o}$,

but they must also be large enough to properly reflect the γ' and β'

distribution. Numerical experimentation will undoubtedly be necessary

to resolve this question.

The system of equations (7.10) can be solved directly as follows.

Consider separately the equations for n=m, m+2, m+4, ... and the

independent set n = m+1, m+3, ... Each set can be written as

$$d_1 \mathcal{N}_1 \quad + a_1 \mathcal{N}_2 \qquad = b_1 \qquad (7.11\text{a})$$

$$a_1 \mathcal{N}_1 \quad + d_2 \mathcal{N}_2 \quad + a_2 \mathcal{N}_3 \qquad = b_2 \qquad (7.11\text{b})$$

$$- \quad - \quad - \quad - \quad - \qquad\qquad\qquad = \quad - -$$

$$a_{k-1} \mathcal{N}_{k-1} + d_k \mathcal{N}_k \quad + a_k \mathcal{N}_{k+1} \qquad = b_k \qquad (7.11\text{c})$$

$$a_{k-2} \mathcal{N}_{k-2} + d_{k-1} \mathcal{N}_{k-1} + a_{k-1} \mathcal{N}_k \qquad = b_{k-1} \qquad (7.11\text{d})$$

$$a_{k-1} \mathcal{N}_{k-1} + d_k \mathcal{N}_k \qquad = b_k \qquad (7.11\text{e})$$

by using an obvious notation. We look for a solution of the form

$$\mathcal{N}_k = -A_k \mathcal{N}_{k+1} + B_k \qquad , \quad k = k-1, k-2, \cdots, 2, 1 . \qquad (7.12)$$

Substituting into the general form (7.11c) produces the coefficient scheme

for k = 2,..., k-1:

$$A_k = \frac{a_k}{d_k - a_{k-1} A_{k-1}} \quad , \qquad (7.13\text{a})$$

$$B_k = \frac{b_k - B_{k-1}}{d_k - a_{k-1} A_{k-1}} .$$

(7.13b)

(7.11a) produces the starting values for this scheme:

$$A_1 = \frac{a_1}{d_1} , \qquad\qquad B_1 = \frac{b_1}{d_1} .$$

(7.13c)

(7.13d)

Having computed A_1, ..., A_{K-1} and B_1, ..., B_{K-1} [which have used all

equations except (7.11e)], we determine ψ_K by using (7.12) and (7.11e),

$$\psi_{K-1} = - A_{K-1} \psi_K + B_{K-1} ,$$

$$a_{K-1} \psi_{K-1} = - d_K \psi_K + b_K ,$$

to determine ψ_K :

$$\psi_K = \frac{b_K - a_{K-1} B_{K-1}}{d_K - a_{K-1} A_{K-1}} .$$

(7.14)

(7.12) then produces successively $\psi_{K-1} , \psi_{K-2} , \cdots , \psi_2 , \psi_1 .$

8. Initial field of divergence

The previous seven sections have been devoted to determining mutually

balanced fields of a streamfunction and geopotential. We now supplement

this with a quasi-geostrophic determination of an initial field of

divergence. The quasi-geostrophic system to be used is one of the simpler

versions defined by Lorenz (Tellus, 1960, p. 364), in which the Coriolis

parameter is fully variable. (The following description is for the

entire globe, but a restriction to the Northern Hemisphere with the

usual symmetry conditions is easily obtained.)

The equations are written in the following nondimensional variables

and symbols:

p = pressure $\div$ (100 cb)

t = time x 2Ω

∇ = horizontal gradient x a

ψ = streamfunction $\div$ $(2\Omega a^2)$

ϕ = variable geopotential $\div$ $(4\Omega^2 a^2)$ (8.1)

Z = $- \ln p$

W = dZ/dt

$\overline{T}$ = standard atmosphere temperature

$\dfrac{\partial X}{\partial p}$ = velocity potential $\div$ $(2\Omega a^2)$, i.e.

$\vec{v}_{div}$ = $\nabla(\partial X/\partial p)$

x = sine (latitude)

The equations consist of the vorticity equation,

$$\nabla^2 \frac{\partial \psi}{\partial t} + \nabla \cdot x \nabla \frac{\partial X}{\partial p} = - \hat{k} \times \nabla \psi \cdot \nabla (x + \nabla^2 \psi) = - A ,$$ (8.2)

the linear balance equation,

$$\nabla \cdot x \nabla \frac{\partial \psi}{\partial t} = \nabla^2 \frac{\partial \phi}{\partial t} ,$$ (8.3)

the simplified first law (combined with the hydrostatic equation),

$$\frac{\partial}{\partial z}\frac{\partial \phi}{\partial t} + S\,W = \frac{\kappa\,q}{8\,\Omega^3 a^2} - \hat{k}\times\nabla\psi\cdot\nabla\frac{\partial\phi}{\partial z} = B, \tag{8.4}$$

the continuity equation,

$$p\,W = \nabla^2 X, \tag{8.5}$$

the upper boundary condition at $p = p_{top}$,

$$W = 0, \quad \nabla^2 X = 0, \tag{8.6}$$

and the bottom boundary condition at $p = 1$:

$$W = \frac{q}{2\Omega\,R\bar{T}}\,w = C. \tag{8.7}$$

In (8.4), S is the static stability function

$$S = S(Z) = \frac{H^2 N^2}{4\Omega^2 a^2} = \left(\frac{R\bar{T}}{g}\right)^2 g\,\frac{d\ln\bar{\theta}}{dz}\,\frac{1}{4\Omega^2 a^2}. \tag{8.8}$$

(S varies from about 0.01 to 0.03.)   q is the heating rate per unit mass.

w in (8.7) is the vertical velocity (dimensional) at the bottom due to

orography and friction:

$$w\,(dim) = 2\Omega\left[\hat{k}\times\nabla\psi\cdot\nabla z_0 + D\,\nabla^2\psi\right]. \tag{8.9}$$

z_0 is the (dimensional) ground height and D is a length $\sim$ 150 meters.

Equation (8.7) should really have a term

$$\frac{4\Omega^2 a^2}{R\bar{T}}\frac{\partial\phi}{\partial t} = \mu\,\frac{\partial\phi}{\partial t}, \quad \mu \sim 10, \tag{8.10}$$

added to the left side. This term is negligible, however, except possibly for the very longest horizontal and vertical wave lengths.

Our goal is to determine $\partial X/\partial p$. We shall assume that the vorticity advection term A in (8.2), the temperature advection and heating term B in (8.4), and the vertical velocity term C in (8.7) are known functions, being determined from the balanced fields of ϕ, ψ and (in the case of w) from the known orographic height.

The bottom of the atmosphere will be set at the constant value $p = 1$ (100 cb), since this uniformity is required by the quasi-geostrophic system. The pressure surfaces on which A and B are most directly computed are set by the standard pressure surfaces on which ϕ and ψ were computed in the variational analysis ($p = 1$, .85, .7, .5, etc.). The logical vertical structure of the variables, however, has both X and W at a series of pressure levels running from $p = 1$ to p_{top}, with the other variables $\partial\psi/\partial t$ and $\partial\phi/\partial t$ at interleaved values of p; this arrangement conflicts with the vorticity advection term being specified at $p = 1$. We therefore assume that A and B can be defined by vertical interpolation at a new convenient set of pressure levels.

The convenient levels are defined by K uniform increments (Δ) in $Z = -\ln p$:

$$Z_k = (k-1)\Delta \; ; \quad k = 1, \ldots, K+1,$$

$$Z_1 = 0,$$

$$Z_{K+1} = -\ln(p_{top}) = K\Delta = Z_{top}$$

$$\Delta = \frac{Z_{top}}{K}$$

(8.11)

K here need not equal the K used in sections 1-7.

At the Z_k levels, the value of p is p_k:

$$p_k = e^{-Z_k} = e^{-\Delta}\, p_{k-1} = r^2 p_{k-1} ,$$

$$r = e^{-\Delta/2} .$$

$$(8.12)$$

Each p_k is a uniform fraction r^2 of the preceding p_{k-1} . $\underline{P_{top}}$, it should be noted, cannot equal zero in this system. The computational convenience of this coordinate system is however great enough to accept this limitation.

At these levels we define the unknowns X_k and W_k and (for $k = 2, \ldots, K$) the known temperature term B_k and stability S_k). Intermediate levels are defined at the average values of Z, and carry with them the unknowns $\partial\psi/\partial t$, $\partial\phi/\partial t$ and the known quantity A. The k subscript is used as shown in this diagram:

$$—\ z_{K+1} \qquad x_{K+1} = w_{K+1} = 0$$

$$\xleftarrow{\cdots}\ \left(\frac{\partial \psi}{\partial t}\right)_K,\ \left(\frac{\partial \phi}{\partial t}\right)_K\ ;\ A_K$$

$$—\ z_K \qquad x_K,\ w_K\ ;\ B_K,\ S_K$$

〜〜〜〜〜〜

$$—\ z_{k+1} \qquad x_{k+1},\ w_{k+1}\ ;\ B_{k+1},\ S_{k+1}$$

$$\xleftarrow{\cdots}\ \left(\frac{\partial \psi}{\partial t}\right)_k,\ \left(\frac{\partial \phi}{\partial t}\right)_k\ ;\ A_k \qquad (8.13)$$

$$—\ z_k \qquad x_k,\ w_k\ ;\ B_k,\ S_k$$

$$\xleftarrow{\cdots}\ \left(\frac{\partial \psi}{\partial t}\right)_{k-1},\ \left(\frac{\partial \phi}{\partial t}\right)_{k-1}\ ;\ A_{k-1}$$

$$—\ z_{k-1} \qquad x_{k-1},\ w_{k-1}\ ;\ B_{k-1},\ S_{k-1}$$

$$—\ z_2 \qquad x_2,\ w_2\ ;\ B_2,\ S_2$$

$$\xleftarrow{\cdots}\ \left(\frac{\partial \phi}{\partial t}\right)_1,\ \left(\frac{\partial \phi}{\partial t}\right)_1\ ;\ A_1$$

$$—\ z_1 \qquad x_1,\ w_1 = C$$

p at the intermediate levels, for example, at the level corresponding to $(\partial \psi / \partial t)_k$, is denoted by

$$r_{k+1/2} = e^{-\frac{1}{2}(z_k + z_{k+1})} = \sqrt{r_k\, r_{k+1}} \qquad (8.14)$$

$$= r\, r_k$$

Our first manipulation is to express (8.4) at k = 2, ..., K with the obvious finite-difference equivalent of $\partial(\partial\phi/\partial t)\partial Z$, apply the operator $\Delta \nabla^2$ to it, and then replace $\partial\phi/\partial t$ by $\partial\psi/\partial t$ from (8.3). This result is (k = 2, ..., K):

$$\nabla \cdot \varkappa \nabla \left[\left(\frac{\partial\psi}{\partial t}\right)_k - \left(\frac{\partial\psi}{\partial t}\right)_{k-1} \right] + \Delta S_k \nabla^2 W_k = \Delta \nabla^2 B_k \, .$$

(8.15a)

If we define the <u>inverse</u> Laplace operator by $\mathcal{L}$, so that (8.5) can be expressed as

$$X = p \, \mathcal{L} \, W,$$

(8.15b)

the vorticity equation can be written (k = 1, ..., K)

$$\nabla^2 \frac{\partial\psi}{\partial t}_k - \nabla \cdot \varkappa \nabla \mathcal{L} \left[\frac{p_{k+1} W_{k+1} - p_k W_k}{\Delta p_{k+1/2}} \right] = -A_k.$$

(8.15c)

We now difference this in the vertical. At the same time we define new variables, so that a symmetric system will emerge. The new variables are

$$V_k = \nu_k W_k,$$

$$U_k = \mu_k \left[\left(\frac{\partial\psi}{\partial t}\right)_k - \left(\frac{\partial\psi}{\partial t}\right)_{k-1} \right],$$

(8.15d)

where v_k and u_k are functions only of k, and are to be determined for our convenience. The result can be written (k = 2, ..., K)

$$\left[\frac{\nu_k}{\Delta \mu_k S_k} \right] \nabla \cdot \varkappa \nabla U_k + \nabla^2 V_k = \frac{\nu_k}{S_k} \nabla^2 B_k,$$

(8.16a)

$$\nabla^2 U_k - \nabla \cdot \mu \nabla \mathcal{L} \left\{ \frac{\mu_k}{\Delta} \left[\lambda \frac{V_{k+1}}{\nu_{k+1}} - (\lambda + \frac{1}{\lambda}) \frac{V_k}{\nu_k} + \frac{1}{\lambda} \frac{V_{k-1}}{\nu_{k-1}} \right] \right\} =$$

$$= - \mu_k \left(A_k - A_{k-1} \right). \tag{8.16b}$$

A convenient choice is

$$\nu_k = \left(\mu_k \, S_k \right)^{1/2} \qquad ; \quad \mu_k = \left(\mu_k / S_k \right)^{1/2} / \Delta . \tag{8.17}$$

This leads to the two equations (k = 2, ..., K)

$$\nabla \cdot \mu \nabla U_k + \nabla^2 V_k = \left(\frac{P_k}{S_k} \right)^{1/2} \nabla^2 B_k , \tag{8.18a}$$

$$\nabla^2 U_k + \frac{\nabla \cdot \mu \nabla \mathcal{L}}{\Delta^2} \left[- \frac{V_{k+1}}{\sqrt{S_k S_{k+1}}} + (\lambda + \frac{1}{\lambda}) \frac{V_k}{S_k} - \frac{V_{k-1}}{\sqrt{S_k S_{k-1}}} \right] =$$

$$\tag{8.18b}$$

$$= - \left(\frac{P_k}{S_k} \right)^{1/2} \frac{A_k - A_{k-1}}{\Delta} .$$

k in these equations runs from 2 to K. In (8.18b), the term V_{k+1} for

k = K vanishes, since $W_{K+1} = 0$. At the lower limit, the V_{k-1} term for

k = 2 is

$$- \frac{\nabla \cdot \mu \nabla \mathcal{L} \, V_1}{\Delta^2 \sqrt{S_1 S_2}} = - \frac{1}{\Delta^2 \sqrt{S_2}} \nabla \cdot \mu \nabla \mathcal{L} \, W_1 . \tag{8.18c}$$

(Note the cancellation of the hitherto undefined number S_1.) Since W_1

is equal to the known quantity C in (8.7), we can remove this term to

the right side of (8.18b) when k = 2.

The bracketed operator on V_k in (8.18b) is now a symmetric tri-diagonal matrix $D_{k\ell}$ $(k,\ell=2,\ldots,K)$

$$D_{k\ell} = \begin{pmatrix} d_2 & -a_2 & 0 & & & & \\ -a_2 & d_3 & -a_3 & & & & \\ 0 & -a_3 & d_3 & & & & \\ & & & & & & 0 \\ & & & & -a_{K-2} & d_{K-1} & -a_{K-1} \\ & & & & 0 & -a_{K-1} & d_K \end{pmatrix} \tag{8.19}$$

$$d_k = \left(\lambda + \frac{1}{\lambda}\right)\frac{1}{\Delta^2 S_k} \quad , \quad a_k = \frac{1}{\Delta^2\sqrt{S_k S_{k+1}}}$$

It will have K-1 positive eigenvalues λ_j and orthonormal eigenvectors ϵ_{kj} $(k,j=2,\ldots,K)$.✓

✓ The determinant of D is equal to $\sinh(K\Delta/2) \div [\Delta^{2(K-1)}\cdot S_2\cdot S_3\cdots S_K\cdot\sinh(\Delta/2)]$, so that λ does not equal zero. The eigenvectors of D do not have the large amplitude variation with k that is present in the variational matrix $B_{\ell k}$ (Table I). In the special case of uniform S_k (i.e., an isothermal atmosphere), the eigenvalues of D are $\lambda_j = (2/\Delta^2 S)\left[\cosh(\Delta/2) - \cos(\pi(j-1)/K)\right]$, and the eigenvectors are simply $\epsilon_{kj} = (2/K)^{1/2}\sin\left[(k-1)(j-1)\pi/K\right]$; $j=2,\cdots,K$; $k=2,\cdots,K$.

We expand the unknowns and the known right sides in these eigen-
vectors as follows:

$$\mu_j = \sum_{k=2}^{K} \epsilon_{kj} U_k \; ,$$

$$\nu_j = \sum_{k=2}^{K} \epsilon_{kj} V_k \; ,$$

(8.20)

$$\beta_j = \sum_{k=2}^{k} \epsilon_{kj} \left(\frac{P_k}{S_k} \right)^{1/2} \nabla^2 \beta_k$$

$$\alpha_j = \sum_{k=2}^{k} \epsilon_{kj} \left(\frac{P_k}{S_k} \right)^{1/2} \left(\frac{A_{k-1} - A_k}{\Delta} \right)$$

$$+ \; \frac{\epsilon_{2j} \nabla \cdot \nu \nabla \chi C}{\Delta^2 \sqrt{S_2}}$$

[The special term in α_j comes from (8.18c).] We thereby arrive at K-1
pairs of two dimensional equations ($j = 1,\ldots,K-1$):

$$\nabla \cdot \mu \nabla \mu_j + \nabla^2 \nu_j = \beta_j \; ,$$

(8.21)

$$\nabla^2 \mu_j + \lambda_j \nabla \cdot \mu \nabla \chi \nu_j = \alpha_j \; .$$

We solve these using spherical harmonics. Before we do so, we note
that the desired result is the field of $(\partial X/\partial p)$--the nondimensional
velocity potential--at the levels in diagram (8.13) corresponding to
$\partial \psi/\partial t$. Using (8.14), (8.15b), (8.15d) and (8.17, we find, in succession,

$$\left(\frac{\partial X}{\partial \varphi}\right)_k = -\left(\frac{1}{\rho}\frac{\partial X}{\partial z}\right)_k = \frac{1}{\Delta}\left[\left(\frac{\rho_k}{r_{k+1}}\right)^{1/2} \mathcal{L} W_k - \left(\frac{\rho_{k+1}}{r_k}\right)^{1/2} \mathcal{L} W_{k+1}\right]$$

$$\tag{8.22}$$

$$= \frac{1}{\Delta}\left[\frac{\mathcal{L} V_k}{\sqrt{S_k \rho_{k+1}}} - \frac{\mathcal{L} V_{k+1}}{\sqrt{S_{k+1} r_k}}\right]$$

It is therefore somewhat more convenient to expand $\mathcal{L} v_j$ rather than

v_j. We now suppress the j subscript for convenience and expand as follows:

$$u_j = \sum_{m,n} u_n^m Y_n^m,$$

$$\mathcal{L} v_j = \sum_{m,n} v_n^m Y_n^m,$$

$$(\alpha, \beta)_j = \sum_{m,n} (\alpha_n^m, \beta_n^m) Y_n^m,$$

$$\tag{8.23}$$

where Y_n^m are the normalized spherical harmonics. By using relation

(7.5), we obtain two equations:

$$- \left[n b_n u_{n-1} + (n+1) b_{n+1} u_{n+1}\right] + n^2 (n+1)^2 v_n = \beta_n,$$

$$n(n+1) u_n + \lambda \left[n b_n v_{n-1} + (n+1) b_{n+1} v_{n+1}\right] = -\alpha_n,$$

where b_n is given by (7.9), and we have suppressed the m superscript.

These can be collapsed into one equation for v:

$$b_n b_{n-1} v_{n-2} + \left[\frac{n b_n^2}{(n-1)} + \frac{(n+1) b_{n+1}^2}{(n+2)} + \frac{n^2(n+1)^2}{\lambda}\right] v_n$$

$$\tag{8.24}$$

$$+ b_{n+1} b_{n+2} v_{n+2} = \frac{1}{\lambda}\left[\beta_n - \frac{b_n}{(n-1)}\alpha_{n-1} - \frac{b_{n+1}}{(n+2)}\alpha_{n+1}\right].$$

This can be solved for v_n by the method sketched at the end of section 7.

αV_k for use in (8.22) is then obtained by reversing (8.20):

$$\alpha V_k = \sum_{j=2}^{K} \epsilon_{kj} \, \alpha v_j.$$

$$= \sum_{j=2}^{K} \epsilon_{kj} \sum_{m,n} v_n^m{}_{(j)} \, Y_n^m(\lambda,\nu) \tag{8.25}$$

The dimensional divergent velocity is then given by

$$\vec{v} \, (dim) \atop div = 2\Omega a \left[\nabla\left(\frac{\partial \chi}{\partial \phi}\right) \right]_{non-dim} \tag{8.26}$$

A final remark concerns the part of $\nabla \cdot \vec{v}$ that is due to friction--the $2\Omega D \, \nabla^2 \psi$ term in (8.9). The boundary treatment underlying (8.9) assumes that frictional effects <u>below p = 1</u> have caused a convergence pattern in a boundary layer below p = 1. The divergence field given by (8.26) does not include this boundary field, only the geostrophic response to it above p = 1. Therefore, when $\vec{v}$(div) from (8.26) is added to the initial field of a primitive equation model (i.e., the goal of section 8), the <u>lowest</u> layer or layers in that model should also have added to them this implied low-level frictional convergence field. If Δp (cb) denotes the (local) thickness of this bottom model layer, the added frictional divergence in that layer should be

$$\nabla \cdot \vec{v} = -\left(\frac{\bar{\rho} \, g}{\Delta p}\right)\left(\frac{p_{sfc}}{100}\right) w,$$

where w is the same horizontal field as that used in (8.7). The

second factor allows roughly for the loss of the lower level part of

(8.26) in regions where p_{sfc} is significantly less than 100 cb.